

Collusion: Exercises Part 1

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Problem 1 (Collusion in a finitely repeated game)

There are two players, $i = 1, 2$. There are also two time periods, $t = 1$ and $t = 2$. In each period, the following “stage game” is played, where player 1 chooses row and player 2 chooses column,

	A	B	C
A	1, 1	0, 0	5, 0
B	0, 0	3, 3	0, 0
C	0, 5	0, 0	4, 4

That is, the players first play this stage game once. Then, after having observed what the rival did in the first round, they play it a second time, after which the overall game is over. Each player maximizes the discounted sum of all his/her payoffs; the discount factor is denoted δ . Assume that $0 < \delta < 1$.

(a) Consider the one-shot game (i.e., the game you get if the stage game is being played only once). Convince yourself that, in this game, there are two (pure strategy) Nash equilibria: (A, A) and (B, B) . Also convince yourself that:

- The (B, B) equilibrium is preferred by both players to the (A, A) equilibrium.
- Each player would be even better off if they could agree to play (C, C) instead, but this is not a Nash equilibrium of the one-shot game.

(b) Now consider the full game. Under what conditions does there exist a subgame perfect Nash equilibrium in which the players choose (C, C) in the first period? You must prove all your claims. In particular, specify the trigger strategies that the players use. Interpret your results and briefly discuss the key assumptions needed for a equilibrium with collusion to exist in a finitely repeated game.

Solution Problem 1

(a) Consider the best responses in the one-shot game. Let $r_2(j)$ denote the best response by player 2 to player 1 choosing j . Inspection of the payoff matrix yields that

$$r_2(A) = \{A\}, \quad r_2(B) = \{B\} \quad \text{and} \quad r_2(C) = \{A\}$$

Similarly, let $r_1(j)$ denote the best response by player 1 to player 2 choosing j . Since the game is symmetric, we also have that

$$r_1(A) = \{A\}, \quad r_1(B) = \{B\} \quad \text{and} \quad r_1(C) = \{A\}$$

Using the best-response functions it is now easy to see that the one-shot game has two Nash equilibria (A, A) and (B, B) . However, it is also trivial to see that both players would prefer the equilibrium (B, B) to the equilibrium (A, A) . At the preferred Nash equilibrium each player obtains the payoff 3. Had they been able to agree to play (C, C) each player would have obtained an even higher payoff of 4; however, (C, C) is not a Nash equilibrium of the one-shot game.

(b) When the stage game is played twice we may be able to sustain an equilibrium in which the players choose (C, C) in the first period.

The first thing to note is that playing (C, C) will never be sustainable in the second period. In the second period, there are no future rounds of the game, so the only equilibrium outcomes that can obtain in the second period are those corresponding to the Nash equilibria of the one-shot game, i.e. (A, A) or (B, B) . The fact that there is more than one possible equilibrium in the second period, with one of the equilibria being preferred by both players, is key to the rest of the solution.

The idea of for how cooperation can be sustained in the first period is as follows: the players threaten to punish non-cooperative first-period behavior by playing the “bad” Nash equilibrium instead of the “good” Nash equilibrium in the second period. In order to make this argument we define a pair of trigger strategies

Denote the action of the row player (i.e. player 1) in period t by x_t , and denote the action of the column player (i.e. player 2) in period t by y_t . Consider then the following pair of trigger strategies:

$$x_1 = C, \quad x_2 = \begin{cases} B & \text{if } x_1 = y_1 = C \\ A & \text{otherwise} \end{cases}$$

$$y_1 = C, \quad y_2 = \begin{cases} B & \text{if } x_1 = y_1 = C \\ A & \text{otherwise} \end{cases}$$

In other words, each player chooses a strategy that involves playing (i) C in the first period and, (ii) B in the second period if and only if both players choose C in the first period (and A otherwise).

Our task is now to check if/when this configuration of strategies constitute a subgame perfect Nash equilibrium (SPNE).

- First we check that no firm has an incentive to deviate unilaterally along the equilibrium path. (Requirement for having a Nash equilibrium.)
- Then we check the same thing off the equilibrium path. (Requirement for SPNE.)

The overall discounted payoff to player i if both players play the trigger strategy:

$$V^{i*} = 4 + 3\delta.$$

Suppose now instead that player i deviates in the first period ($t = 1$), choosing to play A (which is the short-run best response to C). The player's discounted payoff from deviating is

$$V^{i,d} = 5 + \delta$$

since the Nash equilibrium (A, A) will now be played in the second period.

Hence we see that player i will have no incentive to deviate if

$$V^{i*} \geq V^{i,d} \Leftrightarrow 4 + 3\delta \geq 5 + \delta.$$

or, equivalently, if the short-run temptation $(5 - 4) = 1$ is less than the long-term reward from not deviating $\delta(3 - 1)$. This requires that

$$\delta \geq \frac{1}{2}$$

The interpretation is hence the standard one: by deviating, you make a short-term gain but get a lower profit in the second period. So if you're patient enough (sufficiently large δ), then you resist the temptation to deviate.

We also need to check subgame perfection. This involves checking that the strategies involve (subgame perfect) Nash equilibria also off the equilibrium path. In this case this simply involves checking that the actions chosen, according to the strategies, at some first period actions other than (C, C) for a Nash equilibrium.

Hence imagine that we are in a subgame where at least one player did not choose C in period 1. The trigger strategy prescribes that then each player should choose A . We must then verify that this is a Nash equilibrium. Clearly it is: if one player expects the other to play A then the best he can do is to play A as well.

Hence we have shown that we can sustain the outcome (C, C) in period 1 as part of an SPNE of the finitely repeated game if the players care sufficiently much about the future: $\delta \geq \frac{1}{2}$. One should however not that there are other equilibria as well. For example, playing (B, B) in both periods is also an SPNE.

The current example builds on the following general insights.

- We *can* to some extent sustain cooperation as an SPNE also in a finitely repeated game, provided that:
 - The stage game has multiple equilibria.
 - The players agree that the outcome associated with one of the equilibria is better than the outcome of the other.
 - And the players care sufficiently much about the future (just like in an infinitely repeated game).
 - And the players can observe the actions taken by the rival in the previous periods (just like in an infinitely repeated game).

Problem 2 (Problem 6 in Chapter 10 (page 361) of the book by Church and Ware.)

Suppose that demand is given by $p = A - Q$ and that marginal cost is constant and equal to c , where $A > c$. Suppose that there are n firms and the stage game is Cournot.

(a) Find the critical value of the discount factor to sustain collusion if the firms play a supergame and use grim punishment strategies. Assume that the collusive agreement involves equal sharing of monopoly output and profits

(b) How does the minimum discount factor depend on the number of firms? Why?

Solution Problem 2

Let the discount factor be denoted δ (where $0 < \delta < 1$). Our task is to find the critical value of δ , which we can denote δ^{crit} , such that collusion can be sustained if $\delta \geq \delta^{crit}$ given that the players use grim trigger strategies. There are n firms, $i = 1, \dots, n$. We assume equal sharing of profits and output in the collusive agreement.

Consider first the collusive outcome; this is the outcome where the n firms act as a monopolist. Hence let Q denote the total output of the cartel. The total profits of the cartel, we will denote Π , can be written as

$$\Pi = (A - Q)Q - cQ.$$

The output level Q is chosen so as to maximize Π ; the first order condition is

$$-Q + (A - Q) - c = 0,$$

which yields the collusive output level

$$Q^m = \frac{A - c}{2}.$$

Total profits are

$$\begin{aligned}\Pi^m &= (A - c - Q^m)Q^m \\ &= \left(A - c - \left(\frac{A - c}{2}\right)\right)\left(\frac{A - c}{2}\right) \\ &= (A - c)^2 \left(1 - \frac{1}{2}\right) \frac{1}{2} \\ &= \frac{(A - c)^2}{4}\end{aligned}$$

Since the firms share output equally, each firm in the cartel produces

$$q^m = \frac{A - c}{2n}.$$

Similarly, since the firms share profits equally, the profit of each firm is

$$\pi^m = \frac{(A - c)^2}{4n}.$$

Next we consider a firm i that deviates from the collusive agreement. The deviating firm assumes that the other firms will abide by the collusive agreement. It therefore assumes that, if it produces q_i unit of output, total output will be

$$Q = q_i + \sum_{j \neq i} q^m = q_i + (n-1) q^m$$

The profits (in the period of the deviation) for the deviating firm when choosing the output level q_i is hence

$$\pi_i = [A - (n-1) q^m - q_i] q_i - c q_i$$

The optimal deviation, which we will denote q^r , thus satisfies the first order condition

$$-q^r + [A - (n-1) q^m - q^r] - c = 0.$$

Solving for q^r yields

$$q^r = \frac{(A - c) - (n-1) q^m}{2}.$$

But from above we know that $q^m = (A - c) / (2n)$. Using this to substitute for q^m yields

$$\begin{aligned} q^r &= \frac{(A - c) - (n-1) \left(\frac{A-c}{2n} \right)}{2} \\ &= (A - c) \frac{1 - \frac{(n-1)}{2n}}{2} \\ &= \frac{(A - c) (n+1)}{4n} \end{aligned}$$

The profits for the deviating firm (in the period of the deviation) are

$$\begin{aligned} \pi^r &= [A - (n-1) q^m - q^r] q^r - c q^r \\ &= [(A - c) - (n-1) q^m - q^r] q^r \\ &= \left[(A - c) - (n-1) \frac{(A - c)}{2n} - \frac{(A - c) (n+1)}{4n} \right] \frac{(A - c) (n+1)}{4n} \\ &= (A - c)^2 \left[1 - \frac{(n-1)}{2n} - \frac{(n+1)}{4n} \right] \frac{(n+1)}{4n} \\ &= (A - c)^2 \left[\frac{4n}{4n} - \frac{2(n-1)}{4n} - \frac{(n+1)}{4n} \right] \frac{(n+1)}{4n} \\ &= (A - c)^2 [4n - 2(n-1) - (n+1)] \frac{(n+1)}{16n^2} \\ &= \frac{(A - c)^2 (n+1)^2}{16n^2} \end{aligned}$$

We have now characterized the collusive outcome, and the optimal deviation. However, we also need to characterize the Cournot equilibrium since that will serve as the continuation after a deviation. Hence consider the n -firm Cournot equilibrium. Firm i takes the output levels of all the other firms q_j , $j \neq i$, as given. Its profits when it choose output level q_i is hence

$$\pi_i = \left(A - \sum_{j \neq i} q_j - q_i \right) q_i - cq_i$$

where we used that total output is $Q = \sum_{j \neq i} q_j + q_i$ and price is $p = A - Q$. In the Cournot equilibrium firm i set q_i to maximize this profit; the first order condition for profit maximization is

$$-q_i + \left(A - \sum_{j \neq i} q_j - q_i \right) - c = 0.$$

Solving for q_i yields

$$q_i = \frac{(A - c) - \sum_{j \neq i} q_j}{2}$$

Since all firms are identical there will be a symmetric Cournot equilibrium. Hence $q_i = q^c$ for $i = 1, \dots, n$. Moreover, the common output level q^c will satisfy the above first order condition (for firm i). Hence q^c can in this simple symmetric case be solved from

$$q^c = \frac{(A - c) - \sum_{j \neq i} q^c}{2} = \frac{(A - c) - (n - 1) q^c}{2},$$

giving us that

$$q^c = \frac{A - c}{n + 1}.$$

An individual firm's profit in the Cournot equilibrium:

$$\begin{aligned} \pi^c &= [A - nq^c] q^c - cq^c = [A - c - nq^c] q^c \\ &= \left[(A - c) - n \frac{(A - c)}{n + 1} \right] \frac{(A - c)}{n + 1} \\ &= (A - c)^2 \left[1 - \frac{n}{n + 1} \right] \frac{1}{n + 1} \\ &= \frac{(A - c)^2}{(n + 1)^2}. \end{aligned}$$

We can now start to approach the main problem. Thus consider the strategy (the grim trigger strategy) for any given firm i :

- If all firms have played the collusive output ($q_j = q^m$) in all previous periods, play the collusive output ($q_i = q^m$) in this period too.
- If at least one firm did not play the collusive output (some $q_j \neq q^m$) in at least one previous period, play the Cournot-Nash output ($q_i = q^c$).

We want to check when the above strategy allows the firms to sustain the collusive agreement in a subgame perfect Nash equilibrium (SPNE). This involves checking

1. That no firm has an incentive to deviate unilaterally along the equilibrium path.
(Requirement for having a Nash equilibrium.)
2. That no firm has an incentive to deviate unilaterally off the equilibrium path.
(Requirement for subgame perfection.)

Consider firm i 's total discounted profits from any given period $\hat{t}$ if all firms adopt the grim trigger strategy:

$$\begin{aligned} V_i^* &= \sum_{t=\hat{t}}^{\infty} \delta^{t-\hat{t}} \pi^m \\ &= \pi^m \sum_{t=\hat{t}}^{\infty} \delta^{t-\hat{t}} \\ &= \frac{\pi^m}{1-\delta}. \end{aligned}$$

Contrast this with the total discounted profits to firm i from deviating (from the equilibrium path); this yields the short-run profits π^r in the current period but then yields π^c in every subsequent period. Hence

$$\begin{aligned} V_i^d &= \pi^r + \sum_{t=\hat{t}+1}^{\infty} \delta^{t-\hat{t}} \pi^c \\ &= \pi^r + \pi^c \sum_{t=\hat{t}+1}^{\infty} \delta^{t-\hat{t}} \\ &= \pi^r + \frac{\delta \pi^c}{1-\delta}. \end{aligned}$$

This implies that the firm has no incentive to deviate if (and only if)

$$V_i^e \geq V_i^d \Leftrightarrow \frac{\pi^m}{1-\delta} \geq \pi^r + \frac{\delta \pi^c}{1-\delta}.$$

Rearranging this expression yields that it must be that

$$\delta \geq \delta^{crit} \equiv \frac{\pi^r - \pi^m}{\pi^r - \pi^c}.$$

In order to see what this means in terms of the number of firms we need to plug in the expressions for the various profit levels. Using the results above

$$\begin{aligned} \delta^{crit} &= \frac{\pi^r - \pi^m}{\pi^r - \pi^c} \\ &= \frac{\frac{(n+1)^2(A-c)^2}{16n^2} - \frac{(A-c)^2}{4n}}{\frac{(n+1)^2(A-c)^2}{16n^2} - \frac{(A-c)^2}{(n+1)^2}} \quad (\text{where we see that } (A-c)^2 \text{ cancel out}) \\ &= \frac{\frac{(n+1)^2}{16n^2} - \frac{4n}{16n^2}}{\frac{(n+1)^2}{16n^2} - \frac{16n^2}{16n^2(n+1)^2}} \\ &= \frac{(n+1)^2 - 4n}{(n+1)^2 - \frac{16n^2}{(n+1)^2}} \\ &= (n+1)^2 \frac{(n-1)^2}{(n+1)^4 - 16n^2} \\ &= (n+1)^2 \frac{(n-1)^2}{[(n+1)^2 - 4n][(n+1)^2 + 4n]} \\ &= (n+1)^2 \frac{(n-1)^2}{(n-1)^2 [(n+1)^2 + 4n]} \\ &= \frac{(n+1)^2}{(n+1)^2 + 4n}. \end{aligned}$$

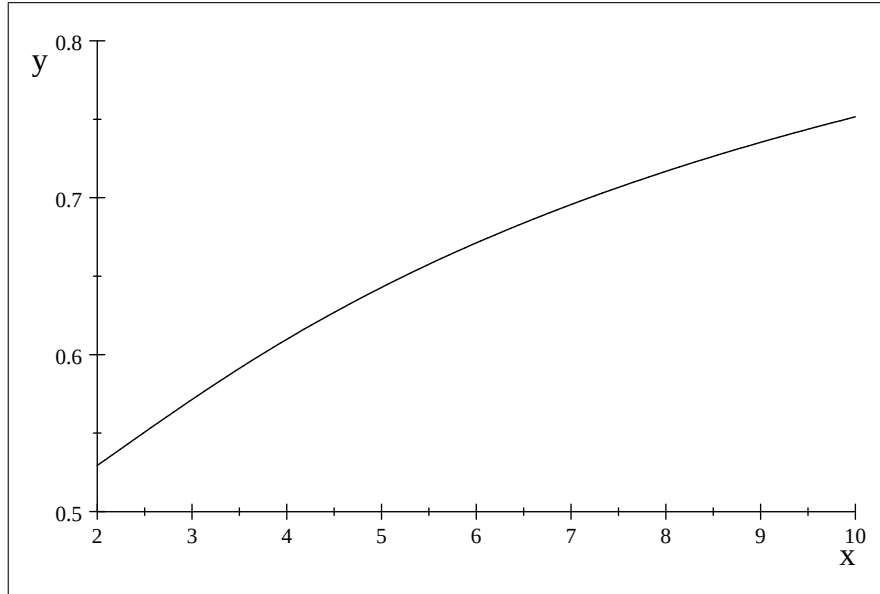
Thus, the critical level of δ , δ^{crit} , is given by

$$\delta^{crit} = \frac{(n+1)^2}{(n+1)^2 + 4n}.$$

- One can also verify that the equilibrium is Nash off the equilibrium path (see book/lecture notes).

We can now consider how the critical discount factor depends on the number of firms

n . Plotting δ^{crit} against n gives the following figure



which shows that δ^{crit} increases in n .

Why is δ^{crit} increasing in n ? Because an individual firm's incentive to deviate from the collusive output is greater when n is larger. One simple way of seeing this is to note that a firm's profits in both the Cournot and the collusive agreement go to zero as $n \rightarrow \infty$; however, the profits associated with a deviation π^r does not (!), instead

$$\lim_{n \rightarrow \infty} (\pi^r) = \frac{(A - c)^2}{16} \lim_{n \rightarrow \infty} \frac{(n + 1)^2}{n^2} = \frac{(A - c)^2}{16}$$

As a consequence $\delta^{crit} \rightarrow 1$ as $n \rightarrow \infty$.

Problem 3 (A specific tax in the collusion problem)

Consider the special case of the previous problem where there are only two firms, $i = 1, 2$. However, suppose that government imposes a specific tax $\tau > 0$ on the product produced by the two firms. This implies that there will be a difference of τ between the price paid by the consumers and the price received by the producers. Argue that the imposition of the tax τ does not affect the sustainability of a collusive agreement since it does not affect the critical discount factor.

Solution Problem 3

To solve this problem we can use the solution to the previous problem. We only need to be careful in defining the price. Let p^c denote the price per unit paid by the consumers and let p denote the price received per unit by the producers. The specific tax implies that $p^c = p + \tau$.

Moreover, it is the consumer price that is relevant in the demand function. Hence demand takes the form

$$p^c = A - Q$$

In terms of the producer price p this implies

$$p + \tau = A - Q$$

or

$$p = A - \tau - Q$$

Hence from the point of view of the firms, the specific tax simply appears as a parallel shift in the demand. Hence if we simply redefine the intercept A to be

$$A(\tau) = A_0 - \tau$$

we can apply the previous analysis simply replace the demand $p = A - Q$ with the more general form

$$p = A(\tau) - Q$$

From the previous analysis we obtained that (with $n = 2$) each firm's profits at the collusive outcome are

$$\pi^m = \frac{(A(\tau) - c)^2}{4n} = \frac{(A(\tau) - c)^2}{8}.$$

Profits in the optimal deviation are (with $n = 2$)

$$\pi^r = \frac{(A(\tau) - c)^2 (n + 1)^2}{16n^2} = \frac{9(A(\tau) - c)^2}{64}$$

and profits in the Cournot equilibrium are

$$\pi^c = \frac{(A(\tau) - c)^2}{(n + 1)^2} = \frac{(A(\tau) - c)^2}{9}.$$

Note that a tax $\tau > 0$ reduces each of the three profit levels.

From the previous exercise we obtain that the critical discount factor was

$$\begin{aligned}
\delta^{crit} &\equiv \frac{\pi^r - \pi^m}{\pi^r - \pi^c} \\
&= \frac{\frac{(A(\tau)-c)^2(n+1)^2}{16n^2} - \frac{(A(\tau)-c)^2}{4n}}{\frac{(A(\tau)-c)^2(n+1)^2}{16n^2} - \frac{(A(\tau)-c)^2}{(n+1)^2}} \\
&= \frac{\frac{(n+1)^2}{16n^2} - \frac{1}{4n}}{\frac{(n+1)^2}{16n^2} - \frac{1}{(n+1)^2}} \\
&= \frac{\frac{9}{64} - \frac{1}{8}}{\frac{9}{64} - \frac{1}{9}} \\
&= \frac{9}{17} \approx 0.53
\end{aligned}$$

Hence, the specific tax does not affect the sustainability of a collusion. The reason is that it affects the short-run gain from a deviation $(\pi^r - \pi^m)$ and the subsequent per period loss $(\pi^r - \pi^c)$ by the same proportion.